M.Sc. 2nd Semester CBCS Scheme w.e.f. 2016-17

Examination, July–2022

MATHS

Paper–16MAT22C2

Measure and Integration Theory

Time allowed : 3 hours] *[Maximum marks : 80*

Note : *Attempt **five** questions in all, selecting **one** question from each unit. **Question No. 9 is compulsory***

Unit–I

1. (a) Prove that outer measure of an interval is equal to its length.

   (b) Show that union of two measurable sets is measurable.

2. (a) Let E be a given set. Then prove that the following statements are equivalent :

   (i) E is measurable.

   (ii) Given $\varepsilon > 0$, there is an open set $O \supset E$ such that $m^*(O - E) < \varepsilon$.

   (iii) There is a G_δ – set $G \supset E$ such that $m^*(G - E) = 0$.

   (iv) Given $\varepsilon > 0$, there is a closed set $F \subset E$ such that $m^*(E - F) < \varepsilon$.

   (v) There is a F_σ – set $F \subset E$ such that $m^*(E - F) = 0$.

   (b) Let $E \subset [0, 1)$ be a measurable set and $y \in [0, 1)$ be given. Then prove that the set $E \dotplus y$ is measurable and $m(E \dotplus y) = m(E)$.

Unit–II

3. (a) Prove that a continuous function defined on a measurable set is a measurable function.

   (b) State and Prove Approximation theorem for simple functions.

4. (a) Let F be a closed subset of R, then prove that a function $g : F \to R$ is coninuous if sets $\{x : g(x) \le a\}$ and $\{x : g(x) \ge b\}$ are closed subsets of F for every rational a and b.

   (b) Let $\{f_n\}$ be a sequence of measurable function which converges to f a. e. on X. Prove that f_n is convergence in measure to f on X.

Unit–III

5. (a) Compare Lebesgue and Riemann Integration.

(b) If f = g a. e. then show that $\int_E f = \int_E g$ and converse is not true.

6. (a) Let f and g be non-negative measurable functions on E. Then prove that for any $\alpha > 0$, $\beta > 0$, $\int_E (\alpha f + \beta g) = \alpha \int_E f + \beta \int_E g$. Moreover, if $f \le g$ on E then $\int_E f \le \int_E g$.

(b) Let f be a non-negative function which is integrable over a set E. Prove that given $\varepsilon > 0$, there exists a $\delta > 0$ such that for every set $A \subset E$ with $m(A) < \delta$, we have $\int_A f < \varepsilon$.

Unit–IV

7. (a) If f is monotonic function on [a, b], then show that it is of bounded variation with total variation $V_a^b(f) = |f(b) - f(a)|$.

(b) If f and g be functions of bounded variation then show that their product is also a function of bounded variation.

8. (a) If f is integrable on [a, b], then prove that the function F defined by $F(x) = \int_a^x f(t)\,dt$ is a continuous function of bounded variation on [a, b].

(b) If f is absolutely continuous function on [a, b], then prove that it is function of bounded variation.

Unit–V

9. (a) Show that outer measure of a null set is zero.

(b) Define Boolean algebra of measurable sets.

(c) Show that every step function is measurable.

(d) Define limit superior and limit inferior of sequence of functions.

(e) Define Lebesgue integral as a integration of simple function.

(f) If m(E) = 0 then show that $\int_E f = 0$.

(g) Define function of bounded variation.

(h) State Fundamental theorem of Integral Calculus.